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EXISTENCE AND UNIQUENESS OF FIXED POINTS ON NON-EXPANSIVE MAPPING IN QUASI-NORMED SPACES

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ABSTRACT

Tulisan ini membahas keberadaan dan ketunggalan titik tetap pada pemetaan non ekspansif dalam ruang bernorma kuasi, yang bertujuan menjamin adanya suatu solusi dari suatu fungsi non ekspansif yang bersifat norma kuasi. Metode penelitian yang digunakan adalah tinjauan literatur dengan memberikan pembuktian serta contoh secara langsung. Penelitian dibahas terlebih dahulu konsep dasar seperti konvergensi, barisan Cauchy, himpunan terbatas, dan kelengkapan dalam fungsi norma kuasi. Selain itu, sifat kompak dan turunannya dijelaskan sebagai bahan pendukung. Pada bagian pemetaan, dijelaskan terlebih dahulu karakteristik operator dalam ruang bernorma kuasi, termasuk pemetaan kontinu dan terbatas, serta ekuivalensi keduanya. Pemetaan non ekspansif dan kontraksi kemudian didefinisikan secara formal, yang menjadi dasar dalam membuktikan eksistensi dan ketunggalan titik tetap. Melalui pendekatan barisan dan sifat kelengkapan, dibuktikan bahwa setiap pemetaan non ekspansif pada ruang kuasi Banach memiliki titik tetap yang tunggal. Di bagian akhir, diperoleh bahwa ruang bernorma kuasi yang bersifat kompak dan konveks menjamin keberadaan titik tetap bagi pemetaan non ekspansif yang didefinisikan pada ruang tesebut.

This study examines the existence and uniqueness of fixed points of non-expansive mappings in quasi-normed spaces, to establish the existence of a solution to a non-expansive function in a quasi-normed space. The research method employed is a literature review, which provides some theorems with proofs and formal examples. The research began by outlining fundamental notions, such as convergence, Cauchy sequences, boundedness, and completeness, in the context of quasi-norms. Furthermore, the properties of compactness and their implications were elaborated as part of a theoretical framework. In the section on mappings, the characteristics of operators in quasi-normed spaces were first explained, including continuous and bounded mappings, along with their equivalence. Non-expansive and contraction mappings were then formally defined, serving as the basis for demonstrating the existence and uniqueness of fixed points. By applying a sequence approach and the completeness property, it was proven that every non-expansive mapping on a quasi-Banach space possesses a unique fixed point. Finally, it was shown that a quasi-normed space that is both compact and convex guarantees the existence of fixed points for non-expansive mappings defined on such spaces.

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INTRODUCTION

The study of mappings within the scope of functional analysis remains a subject of extensive research. The mapping space used also plays a role in producing more general mapping research. Quasi-normed spaces are interesting vector spaces to research, as well as the given mapping, because they are an extension of normed spaces. Quasi-normed spaces provide a generalization of normed vector spaces by relaxing the classical triangle inequality. Specifically, a constant $K \geq 1$ exists such that for all $x, y \in X$, inequality $|x+y|_q \leq K(|x|_q + |y|_q)$ holds, where X is a quasi-normed space with $|\cdot|_q$ as quasi-norm (Firdaus, 2024). This concept has also been investigated in various research directions, including finite-dimensional space settings (Sánchez & González, 2021), operator theory on quasi-normed spaces (Rano & Bag, 2015), and the Banach space properties of operators in quasi-normed settings (Choi et al., 2022).

Some studies use quasi-norm properties in the functions used. This is due to the nature of quasi-norms, which are distinguished by their unique triangle inequality. Research on the concept of function mapping in a quasi-Banach space is explained by Berinde (2024), who explains that approximation and existence results for fixed points of contraction mappings have been established in quasi-Banach spaces. Within the same framework, generalizations of second-type triangle inequalities in quasi-normed spaces have also been explored (Rezaei & Dadipour, 2020). Furthermore, estimates concerning the distance from a given point to the set of fixed points, as well as between two sets of fixed points, have been investigated in quasi-metric spaces (Huu Tron & Thera, 2024). The application of fixed point theory is also an area of growing interest. These applications include studies in the field of dynamical systems (Navascues & Mohapatra, 2024), pramodular structures on complex Cesàro spaces (Bakery & Mohamed, 2022), and graph contractions (Pretușel & Pretușel, 2023).

These previous studies have examined operator properties from different perspectives to explore the extent to which structures can be developed in quasi-normed spaces. It is of interest to examine how far an operator defined on a given space can provide solutions within that space, particularly as this leads to practical approaches for numerical solutions involving specific data and associated functions. This perspective underscores the importance of fixed point existence and uniqueness concerning the defined operator, as highlighted in research on p-vector spaces (Yuan, 2023).

The concept of a fixed point is fundamental. The research and applications mentioned above indicate that the function used is a contraction function. A generalization of the contraction function is a non-expansive function, as explained by Ariza et al. (2014). In other words, this research fills the gap regarding the guarantee of existence and singularity for non-expansive functions in quasi-normed spaces, as a generalization of previous research presented by Proinov (2020).

This paper focuses on fixed points for non-expansive mappings in quasi-normed spaces, presenting a new theorem that offers a novel approach to guaranteeing the existence of fixed points in quasi-normed spaces. The study is inspired by several related topics, including the theoretical foundations of fixed point theory as discussed by Pant et al. (2021) and the fixed point approach to non-expansive mappings by Azam et al. (2023). The initial sections present fundamental definitions from the literature review, including concepts from quasi-normed spaces, convergence, Cauchy sequences, and compactness. Subsequently, the conditions for the existence of fixed points, contraction, and non-expansive mappings on quasi-normed vector spaces are discussed. In the results and discussion section, we establish existence and uniqueness theorems for fixed points in quasi-normed spaces. It is further proven that compact and convex quasi-normed spaces admit fixed points for non-expansive mappings.

METHOD

The method employed in this study involves a literature review of previous research published in scientific journals, providing proof of the related theorem and additional examples. It is systematically explained as follows.

1. Find reference sources that support the research, particularly those related to function mapping in quasi-normed spaces.
2. Identify novelties or gaps in previous research (Berinde, 2024) and use Pant et al. (2021) and Azam et al. (2023) as research ideas, which provide a theorem explaining the generalization of fixed points for a non-expansive function in a quasi-normed space.
3. Explain the theoretical basis related to quasi-norms and the derivatives of defined properties, such as convergence, Cauchy, and compactness, and provide theoretical proofs and related examples.
4. Explain the theoretical basis related to contraction and non-expansive functions and their fixed point requirements in vector spaces with quasi-norms, and provide theoretical proofs and related examples.
5. Provide a discussion through theorems and direct proofs that explain the validity of the singular fixed point of a non-expansive function in a quasi-normed space.
6. Provide a research application in the form of a theorem that guarantees the existence of fixed points of non-expansive functions in quasi-normed spaces that are compact and convex.

RESULTS

Quasi-Normed Space

In this section, basic concepts in vector spaces, including convergence, Cauchy sequences, boundedness, completeness, and compactness, are described. Prior to this study, Rano and Bag (2015) defined a quasi-normed space, along with examples of quasi-normed spaces and an explanation provided by Firdaus (2024).

Definition 1. Let X be the vector space over the field F, and θ be the zero vector in X. A real-valued function $|\cdot|_q$ on X satisfying the following conditions :

1) $|x|_q \geq 0$ for each $x \in X$;
2) $|x|_q = 0$ if and only if $x = \theta$;
3) $|\lambda x|_q = |\lambda||x|_q$ for each $x \in X$ and $\lambda \in \mathbb{R}$;
4) There is such a $K \geq 1$ such that $|x + y|_q \leq K\{|x|_q + |y|_q\}$ for all $x, y \in X$.

called a quasi-norm on X. Pair (X, $|\cdot|_q$) is called a quasi-normed space.

Example 1. $(\mathbb{R}^n, |\cdot|_{q^2})$ with $|x|_{q^2} = \left(\sum_{i=1}^{n} |x_i|^{\frac{1}{2}}\right)^2$ for every $\vec{x} \in \mathbb{R}^n$ is a quasi-normed space.

Proof. It is trivial for (i) – (iii). Next to (iv), for every $\vec{x}, \vec{y} \in \mathbb{R}^n$., we have

$$
\begin{aligned}
|\vec{x}+\vec{y}|_{q^2} &= \left(\sum_{i=1}^{n}|x_i+y_i|^{\frac{1}{2}}\right)^2 \\
&\le \left(\sum_{i=1}^{n}\left(|x_i|^{\frac{1}{2}}+|y_i|^{\frac{1}{2}}\right)\right)^2 \\
&\le 2\left(\sum_{i=1}^{n}|x_i|^{\frac{1}{2}}\right)^2 + 2\left(\sum_{i=1}^{n}|y_i|^{\frac{1}{2}}\right)^2 \\
&\le 2\left(|\vec{x}|_{q^2}+|\vec{y}|_{q^2}\right)
\end{aligned}
$$

There exists $K = 2$ such that $|\vec{x}+\vec{y}|_{q^2} \le K(|x_i|_{q^2} + |y_i|_{q^2})$ for every $\vec{x}, \vec{y} \in \mathbb{R}^n$. In other words, $(\mathbb{R}^n, |\cdot|_{q^2})$ with $|x|_{q^2} = \left(\sum_{i=1}^{n}|x_i|^{\frac{1}{2}}\right)^2$ for $\vec{x} \in \mathbb{R}^n$. It is a quasi-normed space.

The important thing about quasi-norm is that there is $K \ge 1$ such that section (3) of Definition 1 applies. This constant K has properties in quasi-normed spaces described in the following theorem.

Theorem 1. If $(X, |\cdot|_q)$ is a quasi-normed space with constant K, then

$$\left|\sum_{i=1}^{n} x_i\right|_q \le K^{n-1}\left(\sum_{i=1}^{n}|x_i|_q\right) \qquad [2]$$

for all $x_i \in X$ and every $n \in \mathbb{N}$.

Proof. The proof is clear using mathematical induction.

The concepts of convergent, Cauchy, finite, and complete quasi-normed spaces are defined as follows (Firdaus, 2024).

Definition 2. Let (X, $|\cdot|_q$) be a quasi-normed space.

1) A sequence (x_n) of X is said to converge to x if $\lim_{n\to\infty}|x_n - x|_q = 0$ and is said to be Cauchy if $\lim_{m,n\to\infty}|x_n - x_m|_q = 0$.
2) A subset Y of X is said to be bounded if there exists a real positive number M such that $|y|_q \le M$, for all $y \in Y$.
3) A subset B of X is said to be complete if every Cauchy sequence converges in B.

Example 2. The quasi-normed space on Example 1.1 with $n = 2$ is a convergent sequence and also a Cauchy sequence.

Based on Definition 2 above, there is a relationship between the convergent and Cauchy properties, which is explained in the following theorem.

Theorem 2. Let (X, $|\cdot|_q$) be a quasi-normed space.

1) If(x_n) is convergent, then (x_n) is Cauchy on X.
2) If (x_n) has a limit (exists), then its limit is unique.
3) Every sequence (x_{n_j}) of (x_n), where (x_n) converges in X, is a convergent sequence and converges to the same limit of (x_n).

Proof. Part (a) is clear, with implications converging on Definition 1.2. For part (b), it is quite straightforward using the concept of convergence, and part (c) by using the convergence property of (x_n) and the index property of the sequence, which is monotone increasing, then the subsequence (x_{n_j}) converges to the same value as the sequence (x_n).

The convergent and bounded properties are also related in quasi-normed spaces, as explained in the following theorem.

Theorem 3. Let $(X, |\cdot|_q)$ be a quasi-normed space and (x_n) be a sequence in X. If (x_n) converges in X, then (x_n) is bounded in X.

Proof. Suppose (x_n) converges to $x \in X$, such that $\lim_{n\to\infty} |x_n - x|_q = 0$. In other words, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ where $n \geq N$ satisfied $|x_n - x|_q < \epsilon$. Note that for all $n \in \mathbb{N}$, we get $|x_n|_q = |x_n - x + x|_q \leq K(|x_n - x|_q + |x|_q)$ with $K \geq 1$. Choose $\epsilon = 1$, then for $n \geq N$ applies that $|x_n|_q \leq K(1 + |x|_q)$. Now, we set a real constant $M = \max\{|x_1|_q, |x_2|_q, \dots, |x_N|_q, K(1 + |x|_q)\}$. It is clear that $M > 0$, and for all $n \in \mathbb{N}$ we get $|x_n|_q \leq M$.

The compactness properties of quasi-normed spaces are explained in the following definition (Rano & Bag, 2014).

Definition 3. Let $(X, |\cdot|_q)$ be a quasi-normed space. A subset $A \subset X$ is said to be compact if for all sequences (x_n) in A, there exist subsequences that converge to a point in A.

Based on Definition 3 above, there is a theorem that explains the relationship between the properties of completeness and compactness, as stated in the following theorem.

Theorem 4. If subset $A \subset X$ is compact with $(X, |\cdot|_q)$be a quasi-normed space, then A is complete.

Proof. Suppose (x_n) is Cauchy in A, so that $\lim_{m,n\to\infty} |x_m - x_n|_q = 0$. Since A is compact, then (x_n) has a subsequence (x_{nj}) which converges in $x \in A$, i.e $\lim_{n_j\to\infty} \left|x_{n_j} - x\right|_q = 0$. Therefore, we get,

$|x_n - x|_q = \left|x_n - x_{n_j} + x_{n_j} - x\right|_q \leq K\left(\left|x_n - x_{n_j}\right| + \left|x_{n_j} - x\right|_q\right)$. For $n \to \infty$ and $n_j \geq n$ for all $n_j \in \mathbb{N}$, then $\lim_{n\to\infty} |x_n - x|_q \leq K\left(\lim_{n\to\infty} \left|x_n - x_{n_j}\right|_q + \lim_{n\to\infty} \left|x_{n_j} - x\right|_q\right) = 0$. Since a quasi-norm is always non-negative, then $\lim_{n\to\infty} |x_n - x|_q = 0$. Therefore, (x_n) converges to $x \in A$. In other words, A is complete.

The compactness of quasi-normed spaces also has a relationship with closed and bounded properties described in the following theorem.

Theorem 5. Let a subset $Y \subset X$ with $(X, |\cdot|_q)$ be a quasi-normed space. If Y is compact, then Y is closed and bounded.

Proof. Suppose (x_n) in Y. Since Y is compact, there exists a subsequence (x_{n_j}) of (x_n) which converges to $x \in X$. Therefore, (x_n) converges to $x \in X$ based on Theorem 2 parts (ii) and (iii). For the boundedness, since (x_n) converges in Y, then applying Theorem 3 is quite clear.

The mapping from a compact set to a quasi-normed space also preserves its compactness property, as explained in the following theorem.

Theorem 6. A continuous map of a compact set on a quasi-linear normed space is a compact set.

Proof. Suppose $(X, |\cdot|_{qX})$ and $(Y, |\cdot|_{qY})$ are quasi-normed spaces and function $f: P \to Q$ is continuous with $P \subset X$ is compact. Define $f(P) = Q \subset Y$ and take (q_n) in Q. Since f is continuous, there exists (p_n) in P such that $f(p_n) = q_n$ for all $n \in \mathbb{N}$. Since P is compact, then (p_n) has a subsequence (p_{n_j}) which converges to $p \in P$. Applying continuity of f, we have $(f(p_{n_j}))$ converges to $q = f(p) \in Q$. Therefore, Q is compact.

Non-Expansive Mapping on Quasi-Norm Space

In this section, the concept of mapping using operators is given first. The term' operator' used refers to a linear operator, which is commonly known as linear; therefore, we use the word 'operator' only. The explanation begins with the concept of continuous linear operators in quasi-normed spaces as follows (Rano & Bag, 2015).

Definition 4. Let $(X, |\cdot|_{qX})$ and $(Y, |\cdot|_{qY})$ be quasi-normed spaces, D be a subspace of X, and linear operator $T: D \to Y$. Operator T is said to be continuous at $x \in D$ if every sequence (x_n) in D which converges to x, that is $\lim_{n\to\infty} |x_n - x|_{qX} = 0$, then $(T(x_n))$ converges to $T(x)$, that is $\lim_{n\to\infty} |T(x_n) - T(x)|_{qY} = 0$. Furthermore, T is said to be continuous on D, if T is continuous at every $x \in D$.

Example 3. Let $(\mathbb{R}^2, |\cdot|_{q^2})$ and $(\mathbb{R}^3, |\cdot|_{q^3})$ be quasi-normed spaces. Linear operator J with $J: \mathbb{R}^2 \to \mathbb{R}^3$ by definition $J(x) = (x + 2x_2, 3x_1, 2x_2)$ for every $x \in \mathbb{R}^2$ is continuous on $\mathbb{R}^3$. Since the definition of J and quasi norm itself, it is easy to check that if $\lim_{n\to\infty} |x_n - x|_{q^2} = 0$, then $\lim_{n\to\infty} |J(x_n) - J(x)|_{q^3} = 0$.

Rano and Bag (2015) discuss bounded linear operators on quasi-normed spaces as follows.

Definition 5. Let $(X, |\cdot|_{qX})$ and $(Y, |\cdot|_{qY})$ be respectively quasi-normed spaces, D be a subspace of X, and $T: X \to Y$ a linear operator. A linear operator T is said to be bounded if there exists $C > 0$ such that for every $x \in D$, then$|T(x)|_{qY} \leq C|x|_{qX}$.

Example 4. Let $(\mathbb{R}^n, |\cdot|_{q^2})$ for $n = 2$, and the definition of quasi-norm is the same as Definition 1.1. Operator $J: \mathbb{R}^n \to \mathbb{R}^n$ with $J(x) = x$, for all $x \in \mathbb{R}^n$ is a bounded operator. *Proof.* Noted that if we take $C = 1$ such that for all $x \in \mathbb{R}^n$, we have $|J(\bar{x})|_{q^2} \leq C|\bar{x}|_{q^2}$.Therefore, operator J is bounded on $\mathbb{R}^n$ for $n = 2$.

Equivalence of boundedness and continuous properties on a quasi-normed space is provided through the following theorem (Firdaus, 2024).

Theorem 7. Let $(X, |\cdot|_{qX})$ and $(Y, |\cdot|_{qY})$ respectively be quasi-normed spaces, D a subspace of X, and operator $T: D \to Y$ be linear. The following statements are equivalent.

1) Operator T is continuous on D,
2) Operator T is bounded.

Proof. If it is assumed T is unbounded, that is, for every $C > 0$, there is $x_c \in D$ such that $|T(x_C)|_{qY} > C|x_C|_{qY}$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in D$ such that $|T(x_n)|_{q_Y} > n|x_n|_{q_X}$. It is obvious that $x_n \neq 0_D \in D$. If $x_n = 0_D$, so that $T(x_n) = 0_Y$ with $0_Y \in Y$, contradiction. Now, for $n \in \mathbb{N}$, let $y_n = \frac{x_n}{n|x_n|_{q_Y}}$, we get $|y_n|_{q_X} = \frac{1}{n}$. Thus, $\lim_{n\to\infty} |y_n|_{qX} = 0$ and follows that $\lim_{n\to\infty} y_n = 0_D$. Consequently, we get $\lim_{n\to\infty} |T(y_n)|_{qY} = 0$ since T is continuous $0_D \in D$. However, that is contrary to T is continuous, since $|T(y_n)|_{qY} = \frac{1}{n|x|_{q_X}} |T(x_n)|_{qY} > \frac{1}{n|x|_{q_X}} \cdot n|x|_{qX} = 1$. Therefore, the assumption is false, so that T is bounded. For the converse, if T is a bounded operator, there exist $C > 0$ and implies $|T(x)|_{q_Y} \leq C|x|_{q_X}$ for all $x \in D$. Set the sequence (x_n) in D that converges in D, that is $y \in D$, then $\lim_{n\to\infty} |x_n - y|_{qX} = 0$. Therefore, using T is bounded, we get $\lim_{n\to\infty} |T(x_n) - T(y)|_{q_Y} \leq \lim_{n\to\infty} C|x_n - y|_{qX} = C \lim_{n\to\infty} |x_n - y|_{qX}$. From the fact $\lim_{n\to\infty} |x_n - y|_{qX} = 0$, we get $\lim_{n\to\infty} |T(x_n) - T(y)|_{q_Y} = 0$. Thus, linear operator T is continuous on D.

The main discussion is explained in this part. Concept non-expansive mapping of quasi-normed spaces for the proceeding.

Definition 6. Let $(X, |\cdot|_{qX})$ be a quasi-normed space with quasi-constant $K \geq 1$ and operator $T:$ $X \to X$. Operator T is said to be non-expansive if there exists $\alpha \in (0, \frac{1}{K}]$ such that

$$|T(x) - T(y)|_q \leq \alpha |x - y|_q, \quad [3]$$

for every $x, y \in X$.

Example 5. $(\mathbb{R}, |\cdot|_q)$ with quasi norm using Example 1, and operator $T: \mathbb{R} \to \mathbb{R}$ with $T(x) = \frac{1}{5}x$, for all $x \in \mathbb{R}$. Operator T is non-expansive on $(\mathbb{R}, |\cdot|_q)$.

Proof. $|T(x) - T(y)|_q = \left|\frac{1}{5}x - \frac{1}{5}y\right|_q = \frac{1}{\sqrt{5}}|x - y|_q$. Choose $\alpha = \frac{1}{\sqrt{2}}$, such that $|T(x) - T(y)|_q \leq \alpha|x - y|_q$.

The non-expansive mapping property is also continuous, which is described in the following Lemma.

Lemma 1. Every non-expansive mapping of a quasi-normed space $(X, |\cdot|_{qX})$ is continuous.

Proof. It's clear based on non-expansive such that for every (x_n) converges to $y \in X$, then $\lim_{n\to\infty} |T(x_n) - T(y)|_q \leq \lim_{n\to\infty} \alpha|x_n - y|_q = 0$. Therefore, sequence $(T(x_n))$ converges to $T(y)$. Thus, T is continuous.

The following theorem supports the discussion of the non-expansive mapping of a quasi-normed space with a fixed point given below.

Theorem 8. Let $(X, |\cdot|_{qX})$ be a quasi-normed space and $T: X \to X$ be non-expansive. If for every $x_0 \in X$, then $(x_n) = (T^n(x_0))$ is Cauchy.

Proof. Let $x_0 \in$. We set $x_1 = T(x_0)$. Then, we set again $x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$. Thus, for all $n \in \mathbb{N}$ satisfied $x_n = T(x_{n-1}) = T^n(x_0)$. Therefore, for $m, n \in \mathbb{N}$ where $m = n + k$ with $k \in \mathbb{N}$, we get

$$\begin{aligned}
|x_n - x_{n+k}|_{qX} &= |x_n - x_{n+1} + x_{n+1} - x_{n+k}|_{qX} \\
&\leq K\left(|x_n - x_{n+1}|_{qX} + |x_{n+1} - x_{n+k}|_{qX}\right) \\
&\leq K\alpha^n|x_0 - x_1|_{qX} + K|x_{n+1} - x_{n+k}|_{qX} \\
&\leq K\alpha^n|x_0 - x_1|_{qX} + K^2|x_{n+1} - x_{n+2}|_{qX} + K^2|x_{n+2} - x_{n+k}|_{qX} \\
&\leq K\alpha^n|x_0 - x_1|_{qX} + K^2\alpha^{n+1}|x_0 - x_1|_{qX} + K^2|x_{n+2} - x_{n+k}|_{qX} \\
&\leq \sum_{i=0}^{k-1} (K\alpha)^i \left(K\alpha^n|x_0 - x_1|_{qX}\right).
\end{aligned} \quad [4]$$

Since $K \geq 1$ and $\alpha \in (0, \frac{1}{K}]$, so that $K\alpha \leq 1$. Therefore, $|x_n - x_{n+k}|_{qX} \leq K\alpha^n|x_0 - x_1|_{qX}$. If we take $n \to \infty$, hence $\lim_{n\to\infty} |x_n - x_{n+k}|_{qX} = 0$. Thus, $(x_n) = (T^n(x_0))$ is Cauchy.

In this part, an explanation of the existence and uniqueness of non-expansive mapping in quasi-normed spaces is given, using the following theorem as the main result.

Theorem 9. Let $(X, |\cdot|_{qX})$ be a complete quasi-normed space and operator $T: X \to X$. If T is non-expansive, then T has a unique fixed point.

Proof. Suppose $x_0 \in X$, and we set $(x_n) = (T^n(x_0))$. Based on Theorem 8, hence (x_n) is Cauchy. Since X is complete, using Definition 2, then (x_n) converges to $x \in X$ or $\lim_{n\to\infty} x_n = x$. Operator T is non-expansive, so based on Lemma 1, operator T is continuous or $\lim_{n\to\infty} T(x_n) = T(x)$. Note that $(x_n) = T(x_{n-1})$ for all $n \in \mathbb{N}$, so we get $T(x) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} T(x_{n-1}) = \lim_{n\to\infty} x_n = x$. Therefore, operator T has a fixed point. For the uniqueness, assume for $x, y \in X$ are two fixed points for T, then using T is non-expansive, we have$|x-y|_{qX}=|T(x)-T(y)|_{qX}\le\alpha|x-y|_{qX}$. Since $\alpha\in(0,\frac{1}{K})$ where $K\ge 1$, it is clear $|x-y|_{qX}=0$. Therefore, the fixed point is unique.

The application of fixed points on non-expansive mapping in the following quasi-norm space is a condition of a compact and convex set with a quasi-norm that has a fixed point.

Theorem 10. Let $(X,|\cdot|_{qX})$ be a quasi-normed space, and $Y\subset X$ be a compact and convex subset. If operator $T:Y\to Y$ is non-expansive, then $\inf_{y\in Y}|y-T(y)|_{qX}=0$

Proof. Let $r\in Y$, $\alpha\in\left(0,\frac{1}{K}\right]$ with quasi-constant $K\ge 1$ and $T_\alpha:Y\to Y$ where for all $y\in Y$ defined $T_\alpha(y)=(1-\alpha)r+\alpha T(y)$. It is clear that the definition of T_α is well-defined since Y is convex and non-expansive. Furthermore, Y is complete based on Theorem 4. Therefore, using the completeness of Y and the non-expansiveness of T, based on Theorem 9, we get that T has a unique fixed point. In other words, there exists $y_\alpha\in Y$ such that $T_\alpha(y_\alpha)=y_\alpha$, and we have

$$\begin{aligned}|T(y_\alpha)-y_\alpha|_{qX}&=|T(y_\alpha)-T_\alpha(y_\alpha)|_{qX}\\&=|T(y_\alpha)-(1-\alpha)r-\alpha T(y_\alpha)|_{qX}\\&=(1-\alpha)|T(y_\alpha)-r|_{qX}\\&=Q|T(y_\alpha)-r|_{qX}\end{aligned}\qquad[5]$$

with $Q=(1-\alpha)\in(0,\frac{1}{K}]$. If we take $\alpha\to\frac{1}{K}$, maka $Q\to 0$, so we conclude $|T(y_\alpha)-y_\alpha|_{qX}\to 0$. Therefore, for all $y\in Y$, we get $\inf_{y\in Y}|y-T(y)|_{qX}=0$.

DISCUSSION

This study examines theorems that elucidate the existence and uniqueness of fixed points for non-expansive mappings in quasi-normed spaces, employing a proof approach that leverages the properties of sequences. The structure of the quasi-norm is fascinating as it generalizes the concept of a norm, prompting researchers to explore whether properties known for normed spaces can be extended to quasi-normed spaces. Specifically, for fixed points, certain conditions must be met, namely that the mapping is non-expansive, which is a generalization of the contraction function by providing a quasi-constant condition $K\ge 1$ that fulfills the quasi-norm requirements, but does not necessarily fulfill the norm requirements for the triangle inequality. This serves as a bridge for future research to identify more general conditions with another function, thereby providing more flexible guarantees for advantage studies of fixed points in quasi-normed spaces.

Applications of fixed point theory extend to various fields, including nonlinear analysis, topology, differential equations, and game theory. Specifically, for example, the research results can be applied to the Hierarchical Nash Equilibrium Problem (HNEP) to find solutions iteratively using the concept of fixed points on quasi-non-expansive operators (Matsua et al., 2025). In game theory, fixed point iterations on non-expansive functions are used to determine equilibrium solutions (Yufan, 2024).

In general, the existence of fixed points ensures that a function has at least one solution within its domain, making this research a potential link between multiple related scientific disciplines in the future.

CONCLUSION

A quasi-norm is a generalization of a norm that still retains fundamental properties such as convergence, Cauchy sequences, boundedness, completeness, and compactness. This study addresses the existence of a unique fixed point for a non-expansive mapping in a quasi-normed space. The presence of this fixed point guarantees that, for subsequent research, there is a solution to the given function, provided the function is non-expansive. Furthermore, this study can be extended by examining reflexivity properties and considering more varied domains of the space used, such as sequence spaces or more complex function spaces.

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INFORMED CONSENT

The authors have obtained informed consent from all participants.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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