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DEVELOPING A HYPOTHETICAL LEARNING TRAJECTORY FOR LINEAR FUNCTIONS THROUGH COLLABORATIVE MATHEMATICAL MODELING

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ABSTRACT

Students still encounter difficulties in understanding linear functions, particularly when they are required to connect contextual situations with mathematical forms and to understand variables, constant change, and symbolic representations. This study aims to develop and describe students' learning trajectory in learning linear functions through a collaborative mathematical modeling approach using the context of a leaking faucet. This study employed a design research method consisting of three stages: preparing for the experiment, conducting the design experiment, and performing retrospective analysis. The participants were eighth-grade junior high school students grouped heterogeneously according to their mathematical ability. The learning design was structured based on six stages of mathematical modeling. The results indicate that the leaking-faucet context helped students understand the relationship between time and water volume as a linear function. Students were able to identify the problem, formulate assumptions, conduct experiments, construct a model, evaluate the model, and apply it to estimate losses caused by water leakage. Therefore, a learning trajectory based on collaborative mathematical modeling can support students' understanding of linear functions.

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INTRODUCTION

The formal and conceptual characteristics of mathematics often cause students to experience difficulties in learning. In algebra, students' difficulties occur not only in computation but also in mathematization, namely when they must transform information from word problems into mathematical forms. Elagha and Pellegrino (2024) showed that students' errors in solving linear-function problems reflect weaknesses in both conceptual and procedural understanding. Jupri and Drijvers (2016) also showed that students experience difficulties in formulating equations, using variables, and connecting contextual information with algebraic forms. Therefore, an appropriate learning design is needed to address these problems. These findings indicate that learning linear functions should not only emphasize the final algebraic form but also support students in interpreting situations, identifying quantities, and constructing mathematical relationships.

In learning linear functions, students still encounter several difficulties. Linear functions require students to understand relationships among quantities, constant change, and the use of symbolic representations to express these relationships. Recent studies show that students still struggle with context-based problem formulation and interpretation, even though real-life problem solving, instructional methods, and assessment practices have become central issues in mathematics education (Kappassova et al., 2025). In addition, a review by the Ministry of Education and Culture indicates that learning functions requires students to think critically in relating concrete everyday problems to symbolic and graphical forms (Kemendikbud, 2022).

Mathematical modeling is a relevant learning approach because it involves translating real-world situations or problems into mathematical representations, performing symbolic manipulation, and interpreting the results back in the original context (Schukajlow et al., 2015; Leung et al., 2021). By involving students in learning activities through mathematical modeling, students are expected to understand abstract mathematical concepts more meaningfully and recognize the relationship between mathematics and everyday life. Schlüter and Besser (2024) emphasize that authenticity in modelling tasks includes real-world contexts, data, mathematical use, and purpose. In this study, the leaking-faucet context is therefore used not only as an example but also as a basis for helping students construct a linear relationship between time and water volume.

Previous studies also support the use of mathematical modeling in mathematics learning. Özer-Demir & Bukova-Güzel (2024) found that mathematical modeling-based instruction had a positive effect on seventh-grade students' mathematical literacy and academic performance. In addition, Leung et al. (2021) emphasized that modeling tasks should involve authentic and meaningful real-world contexts so that students can experience the relationship between mathematics and reality. Nevertheless, modeling tasks may also be challenging for students because they require them to formulate assumptions, select relevant variables, and validate whether their mathematical results are reasonable in the real situation. Krawitz et al. (2022) explained that making realistic assumptions is an important component of modeling and can become a source of difficulty for students. Therefore, a learning design that uses mathematical modeling needs to be accompanied by support that helps students progress through each stage without reducing their opportunity to construct the model themselves. The need for modeling-based learning is also supported by Chamberlin et al. (2022), who explain that mathematical modeling can facilitate students' sense-making because students are required to connect mathematical ideas with realistic problem situations. In addition, Geiger et al. (2022) highlight that modeling task design needs to consider both task characteristics and pedagogical support so that students can progressively develop modeling competence. These findings strengthen the importance of designing an HLT that guides students through the modeling process rather than presenting linear functions only as symbolic formulas.

Other efforts to overcome students' difficulties in working on mathematical tasks or problems can be carried out through collaborative approaches, the use of media, and scaffolding. Collaborative approaches have been consistently shown to improve students' mathematics achievement (Tasarib & Rambely, 2025). Siller and Ahmad (2024) found that collaborative learning produced a significant increase in mathematics achievement compared with traditional teacher-centered mathematics instruction. Students who learned through a collaborative approach demonstrated better learning outcomes than those taught using traditional methods (Ndebil & Ali, 2024; Sujatha & Vinayakan, 2022). Collaborative learning is also relevant because mathematical modeling involves discussion, negotiation of assumptions, interpretation of data, and agreement on model decisions. Collaborative learning gives students opportunities to exchange ideas, compare strategies, and develop shared understanding. Recent studies have shown that collaborative and cooperative learning can improve students' mathematics achievement and attitudes compared with traditional teacher-centered instruction (Ndebil & Ali, 2024; Siller & Ahmad, 2024). In the context of this study, collaboration is expected to help students work through the modeling process more productively, especially when they need to decide the assumptions, divide experimental tasks, compare calculation results, and agree on whether the model is acceptable.

One form of support that can be integrated into modeling-based learning is scaffolding. Scaffolding is needed because students may not immediately be able to formulate assumptions, organize data, construct a linear function, or interpret the difference between model results and actual data. In mathematical modeling, scaffolding can be provided through guiding questions, confirmation, solution plans, and teacher interventions that are adjusted to students' needs. Siller et al. (2023) showed that scaffolding practices can help students navigate complex modeling tasks in STEM-related contexts, while Gürel (2023) found that teachers' scaffolding methods play an important role during mathematical modeling activities. Greefrath et al. (2022) also emphasized that teachers need pedagogical content knowledge related to modeling tasks and interventions. Solar et al. (2022) found that teacher support for argumentation can be integrated across the mathematical modelling cycle, indicating that discussion and justification are central to students' modeling activities. Thus, scaffolding is important in this study to keep students' thinking aligned with the modeling trajectory while maintaining students' responsibility to construct and evaluate their own mathematical models.

Based on these considerations, this study develops a Hypothetical Learning Trajectory (HLT) for learning linear functions through collaborative mathematical modeling using the context of a leaking faucet. The leaking-faucet context was selected because it is close to students' daily experiences and can represent a linear relationship between time and water volume through a simple experiment. Through this context, students are expected to identify the problem, formulate assumptions, collect and process data, construct a linear function model, evaluate the model, revise it when necessary, and implement the model to estimate losses caused by water leakage. Therefore, the purpose of this study is to develop and describe students' learning trajectory in understanding linear functions through collaborative mathematical modeling supported by teacher scaffolding. This research contributes to the advancement of knowledge regarding the integration of mathematical modeling and collaborative learning in the teaching of linear functions. Practically, the findings are expected to offer educators an empirically grounded learning trajectory and instructional materials that can serve as a reference for designing more effective and contextually relevant mathematics lessons.

METHOD

This study used a design research method based on the framework of Van den Akker et al. (2006). Design research was conducted by designing learning activities that applied a mathematical modeling approach to linear functions through a simple experiment based on the problem of a leaking faucet. The implementation of this research was supported by instruments in the form of a Hypothetical Learning Trajectory (HLT) and student worksheets.

Prior to implementation, the research instruments underwent a validation process to ensure their quality and appropriateness. The HLT and student worksheets were reviewed by two experts in mathematics education and one practicing mathematics teacher. The validation focused on the content validity, pedagogical suitability, and clarity of the instruments. Based on the feedback obtained, revisions were made to refine the learning activities and the guidance provided in the worksheets, ensuring they were aligned with the research objectives and the students' cognitive levels.

The study was conducted in three main phases: (1) preparing for the experiment, (2) the design experiment, and (3) retrospective analysis. The study focused on developing a learning trajectory for linear functions using a mathematical modeling approach. The research began with the formulation of the learning trajectory and conjectures about students' thinking. The designed learning trajectory was then implemented in classroom learning activities, followed by reflection and evaluation of the learning outcomes based on the trajectory that had been developed. The research trajectory is described as a continuous cyclic process, as shown in Figure 1.

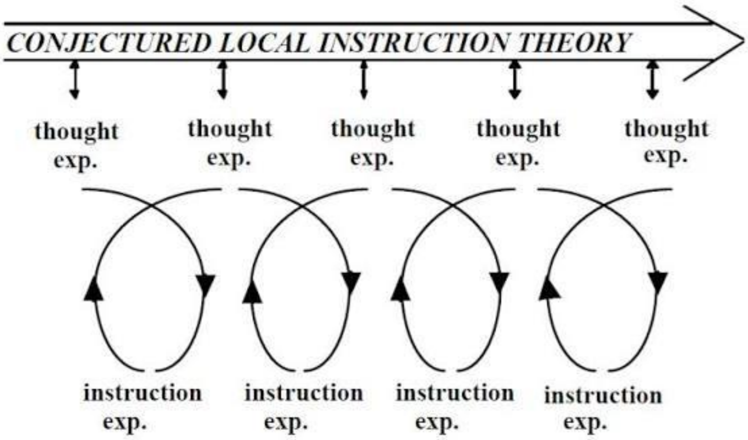


Figure 1. Design Research Cycle

In the preparing for the experiment stage, a literature review was conducted to design the HLT. This stage was carried out to collect the information needed to develop the HLT. After the relevant information had been collected, the HLT was developed. The designed HLT could then be revised based on the results of the learning trial.

Following the HLT formulation, the design experiment was implemented. The design was piloted with three groups of students, each deliberately formed to represent the full spectrum of mathematical proficiency, comprising one high, one medium, and one low mathematical ability student. This purposive selection was chosen because these three groups provided the most complete heterogeneous structure, enabling the researcher to capture the widest range of cognitive interactions and potential learning obstacles within a feasible scope. This approach aligns with the qualitative nature of design research, which prioritizes in depth observation over broad generalizability. The trial was conducted at a school in Sidoarjo. During the lessons, the author served as a non-participating observer, systematically documenting students' verbal and written responses, peer discussions, moments of difficulty, and the teacher's scaffolding strategies, while a regular classroom teacher delivered the instruction to maintain an authentic learning environment and minimize observer interference.

The retrospective analysis stage was conducted to evaluate whether the designed HLT could be implemented properly during the learning process. The learning trajectory at the retrospective analysis stage served as the basis for answering the research questions. This stage aimed to contribute to the refinement of the HLT so that it could better support students' understanding of the learning material. In addition, the HLT was used as a reference for determining the direction and focus of analysis throughout the research process. Various aspects that affected the success of the learning process were examined at this stage. The results of the analysis were then used to draw conclusions and answer the research questions.

RESULTS

The description of the study on linear functions using a collaborative mathematical modeling approach, from the preparation stage to the retrospective analysis stage, is presented as follows.

Preparing for the Experiment

The literature review showed that there are six stages of mathematical modeling according to COMAP & SIAM (2016): (1) identify the problem, (2) make assumptions and identify variables, (3) do the math, (4) analyse and assess the solution, (5) iterate, and (6) implement the model. The learning trajectory was formulated based on these stages of mathematical modeling. Six learning activities were then developed based on these two ideas. The activities started with identifying the problem and ended with applying the learnt mathematical model into context. Students were also required to conduct a simple experiment. It served as a linkage between mathematics and real-world problems, while providing students an opportunity for active participation during learning. The activities conducted by students in the HLT are described below.

Activity 1 (Identify the problem): Through group discussion, students identify causes and impacts of leaking faucet. The goal of this stage is for students to understand the context of the problem that is going to be investigated, which is done by identifying causes and impacts of leaking faucet. Group discussion was also intended to develop students' collaborative skills by exchanging their ideas, voicing their opinions and constructing common grounds of understanding on the problem provided. Students also develop their critical thinking skills by identifying cause-and-effect relationship during this activity. Finally, students also prepared themselves to build a mathematical model after identifying the problem.

Activity 2 (Make assumptions and identify variables): Students formulate assumptions to be used as constraints during the experiment. Students should be able to understand the role of assumptions as constraints to limit the scope of the mathematical model. Assumptions had to be made in such a way that the experiment became more focused on the problem provided. During this activity, students learned how to determine conditions that had to be simplified to build the model. For example, students assumed that volume of leakage was constant or the time needed to be limited when observing the experiment. Making assumptions would help students conduct the experiment and build the mathematical model. Students were also encouraged to discuss what assumptions were needed and had to agree on it collectively.

Activity 3 (Do the math) engaged students in doing an experiment and writing a mathematical model according to patterns in real data. The goal of this activity is to allow students to experience data collection and analysis through conducting a simple experiment involving a leaking faucet. Students gather actual data by measuring volume collected after specific time intervals, allowing students to obtain numerical data on the change in volume over time. Prompted by the pattern of change in volume, students were instructed to find relationship(s) between variables and write a mathematical model using a linear function. Furthermore, this activity was designed to help students learn mathematical modeling ability, analytical thinking, and collaboration skills in discussing and reaching consensus on a mathematical model appropriate for the given problem.

Activity 4 (Analyse and assess the solution) engaged students in analysing how well the solution obtained through the mathematical model corresponds to conditions in the leaking- faucet experiment. Students were guided to think about whether the created mathematical model accurately describes what is happening in the leaking faucet based on this activity. Moreover, this activity allowed students to learn critical thinking skills in assessing model results, identifying potential differences between the model and real data, and determining factors that affect the reasonableness of the applied mathematical model.

Activity 5 (Iterate) required students to review the stages done to solve the problem and modify the mathematical model if needed. The goal of this activity was to enable students to review the mathematical modelling process they performed and improve on it if the product was not consistent with real data according to their group's analysis in activity 4. Students were instructed to iterate by revisiting certain stages in the mathematical modelling process, such as reviewing assumptions, improving data processing methods, or reconstructing the model.

Activity 6 (Implement the model): This activity aims to strengthen students' understanding of linear functions through the application of a mathematical model in the context of losses caused by a leaking faucet over one month. Students are directed to use the linear function model they have constructed to estimate the volume of wasted water and then connect it with information on water-use tariffs to estimate the loss. Through this activity, students not only understand a linear function as a relationship between two variables, namely time and water volume, but also understand how the model can be used to solve real problems. In addition, this activity supports the development of students' mathematical modeling competence and collaborative ability through discussion, task distribution, information searching, and agreement on solutions within the group.

The Design Experiment

In this phase, a learning trial was conducted based on the learning trajectory that had been developed. Before the results of students' learning activities at each stage of mathematical modeling are presented, the selection and grouping of the research participants need to be explained. The trial involved eighth-grade junior high school students. Prior to the trial, a mathematical ability test was administered to 22 students, as two students did not attend the class. The test consisted of five written questions on algebraic operations, which served as prerequisite material for learning linear functions. The test results were then analyzed using descriptive statistics, and students were categorized into high, medium, and low mathematical ability levels based on the mean and standard deviation.

Prior to the design experiment, a mathematical ability pretest was administered to 22 eighth-grade students, as two students were absent from the class. The pretest consisted of five written questions related to algebraic operations, which were considered prerequisite knowledge for learning linear functions. To maintain student confidentiality, the pretest results are presented using student initials/codes only. The results of the mathematical ability pretest are presented in Table 1.

Table 1. Students' Mathematical Ability Pretest Results

No.	Student Initials	Score	Category
1	AS	80	High
2	SQ	25	Medium
3	AA	10	Low
4	AP	35	Medium
5	RD	90	High
6	RH	25	Medium
7	NR	65	High
8	AV	10	Low
9	MI	20	Low
10	SF	40	Medium
11	BC	15	Low
12	AR	5	Low
13	MA	25	Medium

No.	Student Initials	Score	Category
14	HD	30	Medium
15	IV	35	Medium
16	DF	25	Medium
17	NZ	30	Medium
18	NS	35	Medium
19	NB	20	Low
20	SL	40	Medium
21	DA	25	Medium
22	NA	25	Medium

Based on the pretest scores, descriptive statistics were calculated to determine the range of students' mathematical ability categories. The student's mathematical ability categorization, revealing a wide performance spread, directly shaped the formation of heterogeneous groups by ensuring each group contained a mix of cognitive resources to support collaborative modeling. This structure promotes peer scaffolding. This categorization not only underpins the grouping strategy but also influences the interpretation of the research outcomes, as the effectiveness of the learning trajectory is inherently tied to how well these heterogeneous dynamics are managed. The descriptive statistics showed that the minimum score was 5, the maximum score was 90, the mean score was 39.70, and the standard deviation was 19.48, as presented in Table 2.

Table 2. Descriptive Statistics of Students' Mathematical Ability Pre-test

N	Minimum	Maximum	Mean	Std. Deviation
22	5	90	39.70	19.48

The categorization of students' mathematical ability was determined using the mean and standard deviation. Students were classified as high-ability students when their score was greater than or equal to 59.3, medium-ability students when their score was greater than 20.3 and lower than 59.9, and low-ability students when their score was lower than or equal to 20.3. The category criteria are presented in Table 3.

Table 3. Criteria for Student's Mathematical Ability Categories

Score Range	Category
Score >= 59.3	High
20.3 < Score < 59.9	Medium
Score <= 20.3	Low

Based on these criteria, the students were grouped into high, medium, and low mathematical ability categories. From the 22 students who had been categorized, nine students were selected as research participants. The selected students were NR, AA, SQ, AV, RD, SF, AP, AS, and AR. They were then arranged into three heterogeneous groups, with each group consisting of one high-ability student, one medium-ability student, and one low-ability student. This grouping was intended to support collaborative interaction during the modeling activities, especially because the preliminary observation indicated that high- and low-ability students tended to interact more easily through students with medium mathematical ability.

Table 4. Grouping of Research Participants

Group	Student Initials	Group Composition
Group 1	AP, AS, and AR	One high-ability student, one medium-ability student, and one low-ability student
Group 2	AV, RD, and SF	One high-ability student, one medium-ability student, and one low-ability student
Group 3	NR, AA, and SQ	One high-ability student, one medium-ability student, and one low-ability student

After the participants had been grouped, the learning trial was implemented according to the six stages of mathematical modeling. The implementation results are described in the following activities.

Activity 1 (Identify the problem): At the identify the problem stage, students identified the leaking faucet problem by indicating its possible causes and impacts. In general, students did not experience major difficulties in determining the causes of leakage because the problem context was close to everyday life. However, some students still understood the impact of leakage in a simple way, for example, by only stating that water was wasted without directly connecting it to economic losses or wasteful water use.

Penyebab Kebocoran	Dampak Yang Ditimbulkan
Pipa yang mengalami kebocoran	Saat akan menyalakan kran, air malah keluar di bagian yang berlubang / mengalami kebocoran
Sambungan pipa yang longgar / tidak rapat	Air terus-menerus menetes dan menyebabkan pemborosan air yang membuat tagihan air banyak
Penyumbatan didalam pipa	Karena air tidak biso tidak dapat mengalir dgn lancar. terdapat resiko retak bahkan pecah. kebocoran menyebabkan pemborosan air dan membutuhkan biaya perbaikan yg cukup besar

Cause of the leak	The resulting impact
Pipes that have leaks	When turning on the tap, the water came out of the hole instead.
loose or poorly sealed pipe connection	Water keeps dripping, causing waste and leading to high water bills.
blockage inside the pipe	Because water cannot flow freely, there is a risk of cracking or even breakage. Leaks result in water wastage and entail significant repair costs.

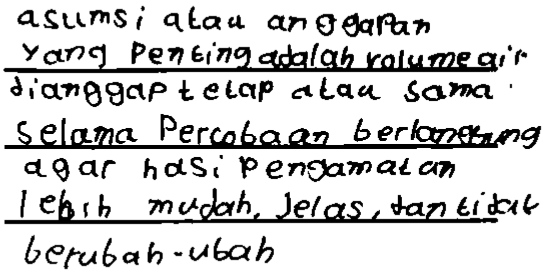
Figure 2. Example of Students' Responses in the Identify the Problem Stage

The teacher's support was provided through guiding questions that encouraged students to connect water leakage with paid water use. The teacher directed students to consider that water used in daily life has a cost, so leakage that is left unattended can cause losses. Students' thinking at this stage showed that they began to understand the leaking-water problem not only as a physical event in the form of dripping water but also as a contextual problem with real impacts.

Activity 2 (Make assumptions and identify variables): At the make assumptions and identify variables stage, students formulated assumptions as constraints so that the water-leakage problem could be modeled mathematically. In general, students understood that water leakage under real conditions is not always constant. However, some students still had difficulty transforming this understanding into clear assumptions to be used in the experiment.

The teacher's support was provided by asking questions that directed students to consider whether a fluctuating leakage condition would facilitate observation and calculation. The teacher then helped students realize that the leakage condition needed to be simplified, for example, by assuming that the leakage rate was relatively constant during the experiment.

Students' thinking showed that they began to understand the function of assumptions in mathematical modeling. Students realized that assumptions are needed to simplify real situations so that experiments can be conducted in a more controlled manner and the data obtained can be used to construct a mathematical model.



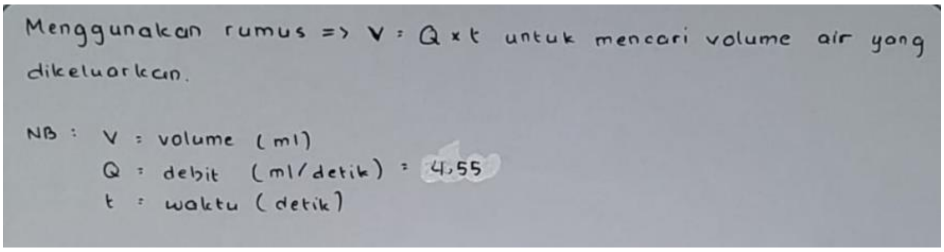
An important assumption is that the water volume remains constant throughout the experiment, ensuring that observations are easier, clearer, and consistent.

Figure 3. Example of Students' Assumption

Activity 3 (Do the math): At the do the math stage, students conducted a water leakage experiment, recorded the water volume at certain time intervals, presented the data in a table, determined the average increase in volume, and formulated a mathematical model in the form of a linear function. In general, students' initial difficulties emerged when determining the observation time interval and understanding cumulative volume recording. In addition, some students were still confused about distinguishing the average increase in volume per interval from the average increase in volume per second.

The teacher's support was provided through confirmation, review of the purpose of the experiment, and guiding questions. The teacher helped students understand that the time interval was used to observe the pattern of increase in water volume, while volume recording was done cumulatively. The teacher also directed students to reread the information requested in the worksheet so that they could adjust their calculation results to the required time unit.

Students' thinking showed that they were able to see the relationship between time and water volume, namely that the longer the observation time, the greater the volume of wasted water. Students then understood that the water volume within a certain time could be obtained by multiplying the average increase in volume per second by time. From this understanding, students were able to formulate a mathematical model by assigning time as a variable.



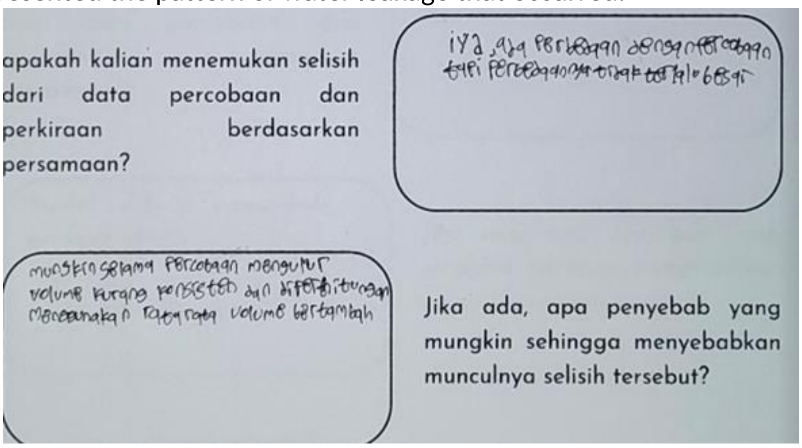
Using the formula V = Q × T to calculate the volume of water discharged.
V = Volume (ml)
Q = debit (ml/second) =4.55
t = time (second)

Figure 4. Example of Students' Mathematical model

Activity 4 (Analyse and assess the solution): At the analyse and assess the solution stage, students compared the predicted results from the mathematical model with the actual data from the experiment. In general, students were able to perform calculations and present the predicted results in a table. The difficulties that emerged were more related to how to interpret the difference between the mathematical model results and the actual data.

The teacher's support was provided by asking students to review the experimental data and compare the results at each interval. The teacher directed students to determine whether the differences that appeared were still reasonable or indicated possible errors in the experimental or calculation process.

Students' thinking showed that they began to understand that a mathematical model is an approximation of a real situation, not a representation that is always exactly identical. Students were able to accept small differences between predictions and actual data as long as the model still adequately represented the pattern of water leakage that occurred.



Did you find a difference between the experimental data and the estimates based on the equation?

(Yes, there is a difference with the experiment but not too far)

(Perhaps during the experiment, the volume measurements were less consistent and the average volume calculation increased)

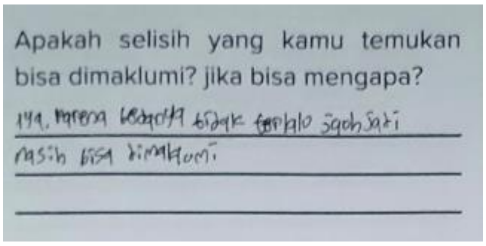
If so, what are the possible causes for the emergence of that discrepancy?

Figure 5. Example of Students' Analysis of the Mathematical Model

Activity 5 (Iterate): At the iterate stage, students evaluated whether the mathematical model they had constructed needed to be improved. In general, students did not immediately repeat the modeling process if the model obtained was considered sufficiently consistent with the actual data. However, when a fairly large difference was found, students began to question the accuracy of the model and needed to review the previous calculation process.

The teacher's support was provided by inviting students to review the worksheet and examine the parts of the calculation that affected the model. The teacher helped students identify possible errors and then directed them to repeat only the necessary parts of the calculation process.

Students' thinking showed that they began to understand that mathematical modeling is not a one-time process, but a process that can be improved if the results are not yet consistent with the real situation. Through the iteration process, students were able to refine the mathematical model so that its predictive results became closer to the actual data.



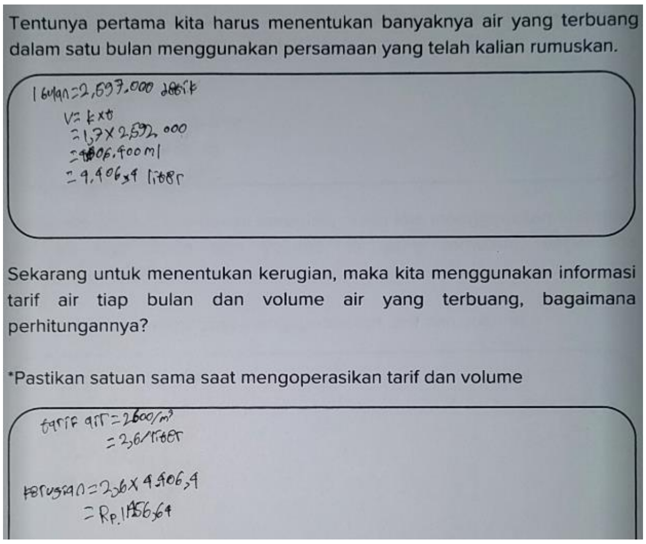
Is the discrepancy you found acceptable? If so, why?

Yes, because the difference is not too far, so it is understandable.

Figure 6. Example of Students' Evaluation of the Mathematical Model

Activity 6 (Implement the model): At the implement the model stage, students applied the mathematical model to solve a real problem, namely estimating losses caused by a leaking faucet over one month. In general, students were able to use the model they had developed to determine the leakage volume over a certain period. However, students experienced difficulties when they had to connect the model with additional information, especially in choosing the appropriate water-use tariff and adjusting different units.

The teacher's support was provided by directing students to consider the context of water use, for example, whether the leakage was viewed in the context of a household or a business. The teacher also reminded students to be careful with differences in units between the water volume in the mathematical model and the water-use tariff. Through guiding questions, students were directed to standardize the units before calculating the loss.



Naturally, the first step is to determine the amount of water wasted in a month using the equation you have formulated.
1 month = 2.592.000 second
V = k x t
= 1.7 x 2,592,000
= 4,406,400 ml
= 4,406.4 liter

Now, to determine the loss, we use the monthly water tariff and the volume of wasted water; how is the calculation performed?
*Ensure the units are consistent when applying rates and volumes.
water rates = 2600 m^3
= 2.6 liter
Loss = 2.6 x 4,406.4
= Rp. 11,456.64

Figure 7. Example of Students' Model Implementation

This stage revealed a notable progression in students' understanding compared to their initial performance at the beginning of the lesson. Initially, students tended to treat the linear function merely as a procedural algebraic exercise, without connecting the equation to the underlying situational context. They also struggled to distinguish between mathematical variables and their physical counterparts, and often overlooked unit conversions when relating the model to real-world quantities. Following the collaborative modeling activities and the teacher's scaffolding, however, students' thinking showed that they were able to use the mathematical model to solve contextual problems more meaningfully. Students did not only calculate the leakage volume but also connected it with the water-use tariff to estimate the loss. At this stage, students began to understand that linear functions can be used to predict the impact of a real problem, thereby demonstrating a shift from procedural execution toward relational understanding and contextual application as envisioned in the developed HLT.

DISCUSSION

Retrospective Analysis

Based on the results of the classroom learning trial, the mathematical modeling process carried out by students showed that the leaking-faucet context could help students understand linear functions more meaningfully. Students did not only learn linear functions as algebraic forms but also used them to represent the relationship between time and the volume of wasted water. In general, the designed learning trajectory was able to facilitate students through the stages of understanding the problem, formulating assumptions, carrying out mathematical processes, evaluating the model, improving the model, and applying the model to a real situation. This is consistent with Borromeo Ferri (2018), who states that mathematical modeling learning needs to provide students with opportunities to experience the modeling process as a whole, from understanding the real situation to evaluating the results of the mathematical model. This finding is supported by Victoria et al. (2021), whose review indicates that mathematical modeling activities can enhance students' learning development by connecting mathematical concepts with real-world situations, while still requiring careful task preparation and classroom support. In this study, the HLT served as a structured design that helped students move from understanding the leaking-faucet context to constructing and using a linear function model.

The findings of this study extend previous research in several notable ways. While earlier studies on HLTs in mathematics education, such as those by Bikner-Ahsbahs et al. (2015), have demonstrated the value of learning trajectories for designing instruction, most have focused on individual student reasoning rather than collaborative group dynamics. Similarly, prior work on mathematical modeling interventions (Stillman & Brown, 2023) has largely emphasized the cognitive aspects of modeling cycles, with less attention to the social and instructional mechanisms that support group-based model construction and refinement. This study contributes to the literature by empirically demonstrating how an HLT, when combined with strategic teacher scaffolding and heterogeneous grouping, can orchestrate collaborative modeling processes in a way that both respects individual thinking and leverages peer interaction. Furthermore, unlike many existing studies conducted in controlled or laboratory settings (Schukajlow et al., 2023), this research was implemented in a natural classroom context, providing evidence of how the HLT functions under authentic instructional constraints.

At the identify the problem stage, students were directed to identify the causes and impacts of a leaking faucet. In general, students were able to understand the causes of leakage because the context was close to everyday life. However, some students initially still understood the impact of leakage in a simple way, namely only as wasted water. Through guiding questions from the teacher, students began to realize that water leakage could also lead to waste and financial loss. This finding indicates that real contexts can serve as a bridge for students to understand the usefulness of mathematics in everyday life. This is in accordance with the views of Borromeo Ferri (2018) and Leung et al. (2021), who state that mathematical modeling activities require students to understand real situations before transforming them into mathematical forms. In addition, Özer-Demir & Bukova-Güzel (2024) also showed that mathematical modeling-based learning can support students in connecting mathematics with real contexts. The use of a leaking-faucet context also reflects the importance of authenticity in modeling tasks. Schlüter and Besser (2024) argue that authenticity involves not only the presence of a real-world context but also the relevance of data, mathematical use, and purpose. In this study, the problem of water leakage became meaningful because students could relate the observed situation to wasted water and financial loss.

At the make assumptions and identify variables stage, students were directed to simplify real situations so that they could be modeled mathematically. Students understood that water leakage under actual conditions is not always constant, but some students still had difficulty formulating assumptions that could be used in the experiment. The teacher provided scaffolding through questions that directed students to consider the need for constraints in observation. After that, students were able to formulate the assumption that the water leakage rate was considered relatively constant during the experiment. This shows that students began to understand the function of assumptions as a form of problem simplification. Krawitz et al. (2022) explain that making realistic assumptions is an important part of mathematical modeling and can also become a source of difficulty. In line with this, Cai et al. (2022) emphasize that open modeling tasks require students to become aware of the need for assumptions so that problems can be solved mathematically.

At the do the math stage, students conducted an experiment, recorded water volume at certain time intervals, presented the data in a table, determined the average increase in volume, and formulated a linear function model. Students' common difficulties emerged when determining the time interval, understanding cumulative volume recording, and distinguishing volume increase per interval from volume increase per second. The teacher provided confirmation and guiding questions so that students understood the relationship among time, cumulative volume, and the rate of increase in volume. After that, students were able to see that the longer the observation time, the greater the volume of wasted water. This understanding became the basis for students to construct a linear function model by assigning time as a variable. This process is in line with Leung et al. (2021), who explain that the stage of working mathematically in modeling involves processing data, using mathematical relationships, and producing mathematical solutions. In addition, Siller et al. (2023) show that the use of a solution plan or work structure can serve as scaffolding that helps students navigate complex modeling tasks.

At the analyse and assess the solution stage, students compared the predicted results based on the mathematical model with the actual experimental data. Students were able to perform calculations and construct comparison tables, but they still needed guidance in interpreting the difference between the model results and the actual data. Some students tended to understand a good model as one that produced values exactly the same as the actual data. The teacher then directed students to understand that a mathematical model is an approximation of a real situation, so small differences can still be accepted if the model is able to represent the data pattern reasonably. Leung et al. (2021) emphasize that validation is an important part of the modeling cycle because students need to reconnect mathematical results with the real context. This is also supported by Borromeo Ferri (2018), who states that students need to learn to interpret and evaluate model results, not merely obtain mathematical answers.

At the iterate stage, students evaluated whether the model they had constructed needed to be improved. When the difference between the model and the actual data was small, students tended to accept the model. However, when there was a large difference, students began to review the calculation process and improve the model with teacher guidance. This process shows that students began to understand mathematical modeling as a cyclic process, not a linear procedure completed in a single attempt. Thus, the iteration process that emerged in learning provides evidence that students did not merely carry out procedures but also began to evaluate and improve the model based on its consistency with the real situation.

At the implement the model stage, students applied the model to estimate losses caused by a leaking faucet over one month. Students' difficulties emerged when choosing the water-use tariff and adjusting the units between the water volume in the model and the tariff unit. Through teacher guidance, students were able to determine the context of water use, standardize units, and use the model to estimate the loss. Thus, students did not only understand linear functions symbolically but also as tools for predicting the impact of real problems. This finding is consistent with Borromeo Ferri (2018) and Leung et al. (2021), who emphasize that mathematical modeling does not stop at model construction but also includes the interpretation and use of the model in real contexts. In addition, Özer-Demir and Bukova-Güzel (2024) show that mathematical modeling-based learning can support students' ability to use mathematics to understand contextual problems. This stage is consistent with Kohen's (2025) view that structured mathematical modeling can maintain the authenticity of an extra-mathematical context while providing enough structure for classroom implementation. The use of water-tariff information required students to extend the linear model beyond the experimental data and use it to make a contextual estimation.

In the implementation of mathematical modeling, the teacher's role remains necessary in validating students' ways of thinking. Although students were able to discuss, determine strategies, and make decisions in groups, in several parts they still needed confirmation to ensure that the steps they chose were aligned with the objectives of modeling. Validation from the teacher was not provided in the form of direct answers but through guiding questions, confirmation of decisions, and direction to review the relationship among the real context, experimental data, and the mathematical model constructed. This is in line with Gürel (2023), who shows that teachers need to provide scaffolding during mathematical modeling activities according to students' needs. Cai et al. (2022) also emphasize that teachers' ability to notice and interpret students' thinking is an important aspect of supporting the modeling process. In addition, Siller et al. (2023) explain that scaffolding in modeling instruction can help students navigate complex tasks without removing students' responsibility for constructing the model. Thus, the teacher plays a role in keeping students' thinking processes on an appropriate trajectory without eliminating students' opportunities to construct models independently.

These findings directly informed our HLT revision. We initially assumed students would navigate validation and iteration with minimal guidance, relying on their algebraic skills. However, students often believed that a good model must match the data perfectly, rather than serving as a reasonable approximation. In response, we embedded guiding questions, such as "How much deviation is still acceptable?" into the validation phase, and restructured the iteration stage to help students differentiate between miscalculations and the inherent limits of linearity. Consequently, the updated HLT frames validation and iteration not as final steps, but as recurring cycles requiring targeted scaffolding. This revision aligns the trajectory more closely with authentic modeling practices and exemplifies how design research iteratively refines theory through classroom evidence.

The importance of the teacher's role is also supported by Barquero and Ferrando (2024), who show that teaching mathematical modeling requires teachers to understand modeling tasks, analyze students' work, and identify constraints during implementation. Çakmak Gürel (2025) further emphasizes that scaffolding in modeling should gradually support learners while transferring responsibility to them. In this study, the teacher's validation therefore functioned not as direct answer-giving but as support for keeping students' thinking aligned with the modeling trajectory. This finding aligns with the broader literature on teacher facilitation in modeling instruction, which underscores the necessity of balancing guidance with student autonomy (Geiger et al., 2022; Siller et al., 2023). The teacher's strategic interventions, particularly through guiding questions and structured prompts, enabled students to navigate the complexities of the modeling cycle while preserving their ownership of the mathematical reasoning process. The findings of this study yield several practical implications for mathematics teachers. The use of contextually grounded, hands-on experiments such as the leaking faucet task proves effective in bridging the gap between students' everyday experiences and formal algebraic representations. Teachers are encouraged to design or adapt similar authentic problems that allow students to physically measure, observe, and connect empirical data to mathematical structures, thereby making abstract concepts such as slope and intercept more tangible.

CONCLUSION

This study emphasizes the effectiveness of a designed mathematical modeling approach, reinforced by strategic teacher scaffolding, in facilitating students' understanding of linear functions. The developed HLT structured the learning process around iterative modeling cycles, enabling students to progressively engage with abstraction, validation, and model refinement. The collaborative setting encouraged peer negotiation of ideas, while targeted scaffolding addressed students' epistemological obstacles particularly in transitioning between contextual situations and formal mathematical representations thereby bridging conceptual gaps without undermining student autonomy. These findings affirm that integrating collaborative modeling with deliberate scaffolding not only enhances conceptual mastery but also cultivates students' appreciation of mathematics as a dynamic problem-solving tool. Ultimately, this design contributes an empirically grounded framework for teaching linear function concepts through authentic, inquiry-based practices, while highlighting the teacher's pivotal role in guiding the modeling process without diminishing students' agency in constructing knowledge.

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INFORMED CONSENT

The authors have obtained informed consent from all participants.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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